NONSTATIONARY ROTATIONAL FLOW OVER A CASCADE OF THIN OSCILLATING AIRFOILS

(NEUSTANOVIVSHIISIA VIKHREVOI POTOK VOKRUG RESHETKI TONKIKH VIBRIRUIUSHCHIKH PROFILEI)

PMM Vol.25, No.5, 1961, pp. 851-857

G. S. SAMOILOVICH
(Moscow)

(Received July 6, 1961)

In [1], the solution of the problem of flow over a cascade of thin oscillating airfoils is investigated with the help of the acceleration potential, which is represented in the form of a series.

Below, an integral representation of the complex acceleration potential is introduced. An analysis of particular cases is not given; this is done in [1], where a bibliography is also given. Here, as a particular case, only the problem of nonstationary rotational flow over a cascade of thin airfoils (a cascade in a vertical gust) is investigated.

1. Let us consider a cascade of thin, slightly cambered airfoils situated in the plane of the complex variable $\zeta = \xi + i\eta$ (Fig. 1). The

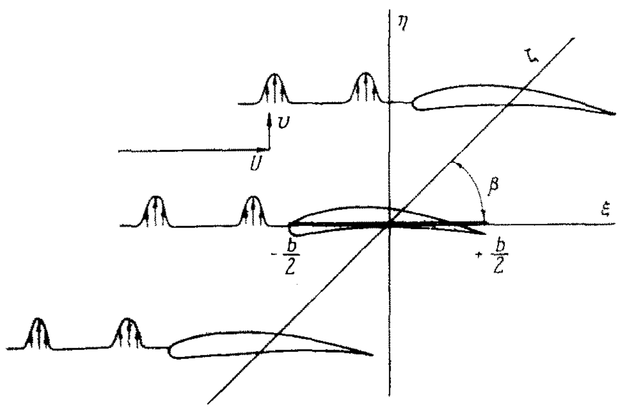

Fig. 1.

axis of the cascade is inclined at the angle β to the axis of the abscissa. The gap is denoted by t, and the chord of the profiles will be taken equal to $b = 2$.

Let us consider the problem of flow of an incompressible fluid approaching with a small angle of attack. The oncoming flow may, in general, be rotational. The airfoils of the cascade may be oscillating with small amplitude.

We shall denote the component of velocity along the ξ-axis at upstream infinity by U and take it to be constant. The velocity component in the η-direction at infinity may be varying with time, but in conformity with the assumption stated above that $v \ll U$ in the flow (with the exception of singular points on the noses of the airfoils).

Then Euler's equations for the nonstationary motion may be linearized and put in the form

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial \xi} = -\frac{1}{\rho}\frac{\partial p}{\partial \xi} = a_\xi, \qquad \frac{\partial v}{\partial t} + U\frac{\partial v}{\partial \xi} = -\frac{1}{\rho}\frac{\partial p}{\partial \eta} = a_\eta \tag{1.1}$$

To Equations (1.1) must be added the continuity equation

$$\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} = 0 \tag{1.2}$$

In these equations, u is the perturbation velocity component, additional to U.

To solve the problem we introduce the complex acceleration potential

$$w(\zeta) = \varphi + i\psi, \qquad a_\xi - ia_\eta = \frac{dw}{d\zeta} \tag{1.3}$$

The acceleration potential is related to the pressure by the obvious relation

$$\varphi = -\rho p + \text{const} \tag{1.4}$$

The constant in this equation may be omitted; p in the equation denotes the perturbation pressure, additional to the value at infinity, upstream of the cascade.

Differentiating the first equation of (1.1), with respect to ξ, and the second with respect to η, adding the resulting expressions and taking into account the continuity equation (1.2), we obtain, as is well known, Laplace's equation $\Delta\phi = 0$.

Differentiating the first equation of (1.1), with respect to η and the second with respect to ξ, and subtracting one expression from the other, we obtain

$$\frac{\partial \Omega}{\partial t} + U\frac{\partial \Omega}{\partial \xi} = 0 \qquad \left(\Omega = \frac{1}{2}\left[\frac{\partial v}{\partial \xi} - \frac{\partial u}{\partial \eta}\right]\right) \tag{1.5}$$

Here Ω is the angular velocity of rotation of velocity particles. The integral of Equation (1.5) must have the following form:

$$\Omega = \Omega\left(\tau - \frac{\xi}{U}\right) \tag{1.6}$$

Equation (1.6) shows that vortices approaching the cascade from infinity, as well as vortices created at the airfoils by changes of circulation, must flow downstream with a velocity U, without change of strength.

Let the airfoils be oscillating and, generally speaking, be subjected to some small deformations. The coordinates of points on an airfoil may be given as a function of the abscissa and time, $f = f(\xi, \tau)$. Then the vertical velocity component on the airfoil, from the condition that there is no flow through the surface, is equal in linearized form to

$$v = \frac{\partial f}{\partial \tau} + U\frac{\partial f}{\partial \xi} \tag{1.7}$$

The boundary value of the vertical component of acceleration, corresponding to the second equation of (1.1), will be

$$a_\eta = \frac{\partial^2 f}{\partial \tau^2} + 2U\frac{\partial^2 f}{\partial \xi \partial \tau} + U^2\frac{\partial^2 f}{\partial \xi^2} \tag{1.8}$$

In solving the problem with the help of the acceleration potential, the boundary condition (1.8) is satisfied in that the derivative $\partial\phi/\partial\eta = a_\eta$ of the function ϕ to be found must have a given value on the airfoil contour. The boundary condition for the velocity (1.7) need be satisfied at only one arbitrary point on the contour, since at the remaining points condition (1.7) will be satisfied automatically (with the satisfying of the boundary condition for the acceleration).

Beside this boundary condition, in flow with circulation it will be required that at the trailing edge the Chaplygin-Joukowski condition be satisfied, which, in the given case, and in view of (1.4), is equivalent to the requirement that the function ϕ be continuous at the trailing edge.

We shall assume the pressure at upstream infinity to be constant, and then $\phi = 0$.

Since the airfoils will be thin and will have small camber, and the amplitude of their oscillation will be small, the boundary conditions on an airfoil contour may be applied at the upper and lower edges of the slit which coincides with the airfoil chord.

The solution of the problem consists of the determination of the nonstationary fields of velocity, acceleration, pressure, vorticity, and the calculation of the forces and moments acting on the airfoils of the cascade.

In view of the linearity of the problem, the fields of the velocities c, accelerations a, pressures p, and vorticity Ω, in the general case of fluid motion, may be regarded as the sum of the corresponding fields created for various reasons. We shall investigate the following fields:

1) Flow field given. The flow moves with the velocity U and carries a given system of free vortices. The velocity field induced by the vortices may be derived in the usual way. The pressure in the fluid will be constant.

2) The field of the perturbations c, a and p, due to steady flow over the cascade of airfoils of given thickness and camber, with velocity U and at the mean angle of attack.

3) The field of the perturbations c, a and p, due to nonstationary flow over the cascade of airfoils of zero thickness and camber. The perturbation field is due to: (a) variations in the velocity of the approaching flow; (b) oscillations and deformations of the airfoils; and (c) the influence of the vortex sheet trailing from the airfoils in nonstationary flow.

Since ϕ must satisfy Laplace's equation, conformal transformations may be introduced and the problem solved in a parametric plane. With the aid of the function

$$\zeta = \sin\beta z - \frac{i}{q}\frac{\cos\beta}{\cosh q}\ln\left(\cosh qz + \sqrt{\sinh^2 qz - \sinh^2 q}\right)\cosh^{-1} q \qquad \left(q = \frac{\pi b}{2t}\right) \tag{1.9}$$

we transform the cascade plane $\zeta = \xi + i\eta$ into the parametric cascade plane $z = x + iy$.

Corresponding to the cascade of slits in the ζ-plane, there will be a cascade of slits in the z-plane, but without stagger ($\beta = \pi/2$). The length of the slits will still be equal to $b = 2$, and the gap to t.

The condition $d\zeta/dz = 0$ gives the points in the z-plane which correspond to the edges of the slits in the ζ-plane:

$$\sinh qx_{1,2} = \pm\sin\beta\sinh q \tag{1.10}$$

2. We shall find the expression for the complex acceleration potential in the z-plane outside the cascade of slits, which are not staggered ($\beta = \pi/2$). Let the y-axis be the axis of the cascade and the origin of coordinates be the center of one of the slits.

In [1] it was shown that the complex acceleration potential in this case may be represented as a series*

$$w = \sum_{n=1}^{\infty} A_n \left[\frac{1}{q} \ln \left(\cosh qz + \sqrt{\sinh^2 qz - \sinh^2 q}\right) - z\right]^n + iB\sqrt{\frac{\sinh q\,(z-1)}{\sinh q\,(z+1)}} \quad (2.1)$$

$$q = \pi b / 2t$$

Here A_n and B are constant (with respect to z) quantities. Then the complex acceleration is given by the series

$$a = \left(\frac{\sinh q z}{\sqrt{\sinh^2 qz - \sinh^2 q}} - 1\right) \sum_{n=1}^{\infty} nA_n \left[\frac{1}{q} \ln \left(\cosh qz + \sqrt{\sinh^2 qz - \sinh^2 q}\right) - z\right]^{n-1} +$$

$$+ \frac{iqB \sinh 2q}{2 \sinh q\,(z+1) \sqrt{\sinh q\,(z-1) \sinh q\,(z+1)}} \quad (2.2)$$

The last term of (2.2) becomes real on the slits. The coefficients A_n are determined by the known boundary values of the imaginary part of the complex acceleration on the edges of the slits.

At infinity, the function represented in (2.2) by the series is of order $1/\sinh^2 qz$. On the upper and lower edges of the slits the imaginary parts of this function takes on equal values, equal to the boundary value of the normal acceleration. The real parts of the function of the complex acceleration are equal in magnitude but opposite in sign. For representing this component of the complex acceleration, use can be made of the method used in thin-wing theory [2].

---

* In that case, if the z-plane is taken to be parametric, the real part of the complex potential must be equal to zero at the point $\sinh qx = \sin\beta \sinh q$, corresponding to the trailing edge in the ζ-plane. Then the last term in (2.1) must be

$$\frac{\sqrt{1 + \sin^2 \beta \sinh^2 q}\, \sinh qz + \sin \beta \sinh q \cosh qz}{\sqrt{\sinh^2 qz - \sinh^2 q}}$$

A periodic analytic function, tending toward zero at upstream and downstream infinity, may be represented in the cascade plane by an integral which is obtained from Cauchy's integral

$$F(z) = \frac{q}{2\pi i} \int_L F(\zeta) \coth q(\zeta - z)\, d\zeta \tag{2.3}$$

Here the integration is around one of the slits. Let us consider the function

$$\Phi(z, \tau) = a(z, \tau) \sqrt{\sinh^2 qz - \sinh^2 q} \tag{2.4}$$

This function is periodic and approaches zero at infinity. On the upper and lower edges of the slits the real part of this function, equal to $a_y(x, \tau)\sqrt{(\sinh^2 q - \sinh^2 qx)}$, takes on values which are equal in magnitude and opposite in sign. The imaginary part, equal to $a_x(x, \tau)$ $\sqrt{(\sinh^2 q - \sinh^2 qx)}$, takes on equal values at corresponding points of the two edges.

Then, using the integral (2.3) to represent the function $\Phi(z, \tau)$, and taking a slit to be the contour of integration, we obtain instead of the series in (2.2) one integral representation of the function in terms of the known boundary values of the normal acceleration. Leaving out the intermediate steps, we obtain the final expression for the complex acceleration

$$a(z, \tau) = \frac{q}{\pi i} \frac{1}{\sqrt{\sinh^2 qz - \sinh^2 q}} \int_{-1}^{+1} a_y(\xi, \tau) \sqrt{\sinh^2 q - \sinh^2 q\xi} \coth q(\xi - z)\, d\xi + \\ + \frac{iqB \sinh 2q}{2 \sinh q(z+1) \sqrt{\sinh q(z-1) \sinh q(z+1)}} \tag{2.5}$$

The complex acceleration potential w is found by integrating (2.5) along an arbitrary curve from $z = +1$, where the real part of w may be put equal to zero, and the immaterial imaginary constant can be thrown away.

$$w(z, \tau) = \frac{q}{\pi i} \int_{+1}^{z} \frac{dz}{\sqrt{\sinh^2 qz - \sinh^2 q}} \int_{-1}^{+1} a_y(\xi, \tau) \sqrt{\sinh^2 q - \sinh^2 q\xi} \coth q(\xi - z)\, d\xi + \\ + iB \sqrt{\frac{\sinh q(z-1)}{\sinh q(z+1)}} \tag{2.6}$$

Thus the complex acceleration potential consists of a term which depends only on the normal acceleration on the contour, and a term which is determined by the kinematic condition of the motion by integrating (1.1).

For example, in the case of a harmonically oscillating process (oscillation of the airfoils or flow of periodic vortex wakes over the cascade) the velocity and acceleration of the flow may be written in the form

$$v'(z, \tau) = v(z) \exp j\omega\tau, \qquad a'(z, \tau) = a(z) \exp j\omega\tau$$

Here ω is the frequency of the oscillatory process, j is the imaginary integer, not the same as the imaginary integer i.

Then integration of (1.1) under the condition that the velocity far ahead of the cascade is equal to U, and the normal velocity on the leading edge of the airfoil is equal to $v_0 \exp j\omega\tau$, gives

$$v_0 U e^{j\omega\tau} = e^{jk} \int_1^\infty a(-z)\, e^{-jkz}\, dz \tag{2.7}$$

Here $k = \omega b / 2U$ is the Strouhal number.

The further solution of the problem proceeds along usual lines [1]. From (2.7) and (2.5), B can be determined. In Equation (2.6), separating the real part (with respect to i), we find the distribution of pressure. Integrating the distribution of pressure around the airfoil contours, we determine the nonstationary lift force.

3. In the case of steady flow over the cascade, the normal acceleration on the contour will have only the convective term, which, according to (1.8), is equal to

$$a_y(x) = U^2 \frac{d^2 f}{dx^2} = KU^2 \tag{3.1}$$

Here $K = K(x)$ is the airfoil camber. The camber being known, (2.5) makes it possible to find the acceleration field, and, with the help of (2.7), putting $\omega = k = 0$, to find the coefficient B. These transformations can be carried out in general form.

It is also possible to generalize the problem to the case where the normal accelerations on the upper and lower edges of the slits are not equal (the airfoils have different cambers on the convex and concave sides).

4. In the case of nonstationary vortex flow, the cascade may be taken to be made up of airfoils having zero thickness and camber (a cascade of plates), since the influence of thickness and camber can be attributed to stationary flow.

Let us consider a cascade of airfoils with zero stagger ($\beta = \pi/2$) over which vorticity waves flow periodically, coming from infinity with constant velocity U, their front being perpendicular to the flow direction.

Consider the given fields in the absence of the cascade. Let the vertical velocity vary as in gusts, traveling with velocity U. The form of the gusts is given by the function

$$v = v\left(\tau - \frac{x}{U}\right) \tag{4.1}$$

These gusts are produced by traveling vorticity waves

$$\Omega = \frac{1}{2}\frac{\partial v}{\partial x} \tag{4.2}$$

In accordance with the second of equations (1.1), the vertical accelerations in the flow are equal to zero:

$$a_y = \frac{\partial v}{\partial \tau} + U\frac{\partial v}{\partial x} \equiv 0 \tag{4.3}$$

We assume the pressure field to be a constant everywhere. If a cascade is placed in the flow it will produce disturbances, since the airfoils are impenetrable, and the normal component of velocity on them must be equal to zero.

The complex acceleration potential of the perturbed flow is, from (2.6), equal to ($a_y \equiv 0$):

$$w(z, \tau) = iB\sqrt{\frac{\sinh q(z-1)}{\sinh q(z+1)}} \tag{4.4}$$

The coefficient B is determined from the condition that the vertical perturbation velocity on the airfoil is equal to $-v(\tau - x/U)$.

Let us investigate the case of a vertical gust, when the velocity varies according to a harmonic law

$$v = v_0 \exp j\omega\left(\tau - \frac{x}{U}\right) \tag{4.5}$$

The general case can then be obtained by representing the gust by a Fourier series.

Using (2.7), with the acceleration determined from the complex potential (4.4), we obtain

$$B = v_0 U e^{j\omega\tau} R(k, q) \qquad \left(R(k, q) = \frac{e^{jk}}{1 + jke^{jk} I(k, q)}\right) \tag{4.6}$$

Here [1]

$$I\ (k,\ q) = \int_1^{\infty} \left(\sqrt{\frac{\sinh q\ (x+1)}{\sinh q\ (x-1)}} - e^q \right) e^{-jkx} dx \tag{4.7}$$

The pressure field is then found by separating the real part (with respect to i) of (4.4). In particular, the distribution of pressure on an airfoil of the cascade is given by the expression

$$p = \rho B \sqrt{\frac{\sinh q\ (1-x)}{\sinh q\ (1+x)}} \tag{4.8}$$

Note that the pressure-distribution law does not depend (in the linearized problem) on the form of the gust and the frequency of the approaching vorticity waves.

The nonstationary lifting force acting on an airfoil of the cascade is found by integrating (4.8) over the airfoil contour:

$$L = -4\rho B \frac{t}{b} \sinh \frac{\pi b}{2t} \tag{4.9}$$

In the particular case of oscillation of a single wing ($q = 0$) the integral (4.7) is expressed by means of Hankel functions

$$I\ (k,\ 0) = \int_1^{\infty} \left(\sqrt{\frac{x+1}{x-1}} - 1 \right) e^{-jkx}\, dx = -\frac{\pi}{2} [H_1^{(2)}(k) + jH_0^{(2)}(k)] - \frac{1}{jk} e^{-jk}$$

The function (4.6) becomes the function

$$R\ (k,\ 0) = \frac{2j}{\pi k\ [H_1^{(2)}(k) + jH_0^{(2)}(k)]} \tag{4.10}$$

and gives the solution for a wing in a sinusoidal vertical gust which was obtained by Sears [3].

BIBLIOGRAPHY

1. Samoilovich, G.S., Obtekanie aerodinamicheskoi reshetki tonkikh vibriruiushchikh profilei (Flow over an aerodynamic cascade of thin, oscillating airfoils). ***PMM*** Vol. 25, No. 4, 1961.

2. Sedov, L.I., ***Ploskie zadachi gidrodinamike i aerodinamike (Plane Problems of Hydrodynamics and Aerodynamics)***. Gostekhizdat, 1950.

3. Sears, W.R., Some aspects of nonstationary airfoil theory and its practical application. *Jour. Aero. Sci.* Vol. 8, No. 2, 1941.

Translated by A.R.